Final Exam − Review − Problems

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Note: In all the problems below, V is a finite-dimensional inner-product space (except in problems 1 and 7(a)-(d), where V is just a finite-dimensional vector space)

Problem 1:

Let U and W be subspaces of a vector space V, with $dim(U) \geq dim(W)$. Show that there exists $T \in \mathcal{L}(V)$ such that $T(U) = W$.

Problem 2:

Suppose $T \in \mathcal{L}(V)$ satisfies $< T(e_i), e_j >= 0$ if $i \neq j$ and 1 otherwise (for all i and j). Calculate $\mathcal{M}(T)$.

Problem 3:

Let T and S be self-adjoint operators on V such that $TS = ST$. Show that there exists an orthonormal basis of V whose elements are eigenvectors of *both* S and T (that is, S and T are *simultaneously diagonalizable*)

Problem 4:

In the following V^* denotes the set of all linear functionals on V[1], and given v, $\phi_v \in V^*$ denotes the functional $\phi_v(u) =< u, v >$.

Define $\Phi : V \longrightarrow V^*$ by: $\Phi(v) = \phi_v$

Show that Φ is an isomorphism of vector spaces!

[1] that is, the set of linear transformations from V to $\mathbb{F}$

Problem 5:

Let U be a subspace of V, and P be the orthogonal projection on U. Let $J : U \longrightarrow V$ denote the inclusion map, that is, $J(u) = u$. Show that $J^* = P$

Problem 6:

Let V be an inner-product space and W be any vector space, and $T \in \mathcal{L}(V, W)$. Given $w \in W$, define $S_w = \{v \in V \mid T(v) = w\}$ (the set of vectors in V that map to W). Show that the smallest element $\hat{w}$ of S_w(if it exists)[2] is orthogonal to any vector $Nul(T)$

Problem 7: TRUE/FALSE EXTRAVAGANZA!!!

(a) If U, W, Z are subspaces of V, and $dim(V) = dim(U) + dim(W) + dim(Z)$, then $V = U \oplus W \oplus Z$

(b) If W is a fixed subspace of V, then $\{T \in \mathcal{L}(V) \mid W \text{is a } T\text{-invariant subspace of} V\}$ is a subspace of $\mathcal{L}(V)$

(c) If $T, S \in \mathcal{L}(V)$, and S is invertible, then T and STS^{-1} have the same eigenvalues, including multiplicities

(d) If $V = \mathbb{R}^2$ and $T^2 = T$, then there is a basis of V consisting of eigenvectors of T

(e) If $T = S^*S$ for $S \in \mathcal{L}(V)$, then all the eigenvalues of T are nonnegative

(f) If $\mathbb{F} = \mathbb{C}$, and T is normal and nilpotent, then $T = 0$

(g) If $\mathbb{F} = \mathbb{C}$, and $\|Tx\| = \|x\|$ for all x, then there is a basis of V consisting of eigenvectors of T.

[2] By this we mean that if u is any other vector in S_w, then $\|\hat{w}\| \leq \|u\|$